HEINZ-SCHWARZ INEQUALITIES FOR HARMONIC MAPPINGS IN THE UNIT BALL

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ABSTRACT. We first prove the following generalization of Schwarz lemma for harmonic mappings. Let u be a harmonic mapping of the unit ball onto itself. Then we prove the inequality $\|u(x) - (1 - \|x\|^2)/(1 + \|x\|^2)^{n/2}u(0)\| \leqslant U(|x|N)$. By using the Schwarz lemma for harmonic mappings we derive Heinz inequality on the boundary of the unit ball by providing a sharp constant C_n in the inequality: $\|\partial_r u(r\eta)\|_{r=1} \geqslant C_n$, $\|\eta\| = 1$, for every harmonic mapping of the unit ball into itself satisfying the condition $u(0) = 0$, $\|u(\eta)\| = 1$.

1. INTRODUCTION

E. Heinz in his classical paper [4], obtained the following result: If u is a harmonic diffeomorphism of the unit disk $\mathbf{U}$ onto itself satisfying the condition $u(0) = 0$, then

$$|u_x(z)|^2 + |u_y(z)|^2 \geqslant \frac{2}{\pi^2},\ z \in \mathbf{U}.$$

The proof uses the following representation of harmonic mappings in the unit disk

$$u(z) = f(z) + \overline{g(z)}, \tag{1.1}$$

where f and g are holomorphic functions with $|g'(z)| < |f'(z)|$. It uses the maximum principle for holomorphic functions and the following sharp inequality

$$\liminf_{r\to 1^-} \left|\frac{\partial u(re^{it})}{\partial r}\right| \geqslant \frac{2}{\pi} \tag{1.2}$$

proved by using the Schwarz lemma for harmonic functions. The aim of this paper is to generalize inequality (1.2) for several dimensional case.

If u is a harmonic mapping of the unit ball onto itself, then we do not have any representation of u as in (1.1).

It is well known that a harmonic function (and a mapping) $u \in L^\infty(B^n)$, where $B = B^n$ is the unit ball with the boundary $S = S^{n-1}$, has the

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following integral representation

$$u(x) = \mathcal{P}[f](x) = \int_{S^{n-1}} P(x,\zeta) f(\zeta) d\sigma(\zeta), \tag{1.3}$$

where

$$P(x,\zeta) = \frac{1-\|x\|^2}{\|x-\zeta\|^n}, \zeta \in S^{n-1}$$

is Poisson kernel and σ is the unique normalized rotation invariant Borel measure on S^{n-1} and $\|\cdot\|$ is the Euclidean norm.

We have the following Schwarz lemma for harmonic mappings on the unit ball B^n (see e.g. [1]). If u is a harmonic mapping of the unit ball into itself such that $u(0) = 0$ then

$$\|u(x)\| \leqslant U(rN), \tag{1.4}$$

where $r = \|x\|$, $N = (0, \dots, 0, 1)$ and U is a harmonic function of the unit ball into $[-1, 1]$ defined by

$$U(x) = \mathcal{P}[\chi_{S^+} - \chi_{S^-}](x), \tag{1.5}$$

where χ is the indicator function and $S^+ = \{x \in S : x_n \geqslant 0\}$, $S^- = \{x \in S : x_n \leqslant 0\}$. Note that, the standard harmonic Schwarz lemma is formulated for real functions only, but we can reduce the previous statement to the standard one by taking $v(x) = \langle u(x), \eta\rangle$, for some $\|\eta\| = 1$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. Indeed, we will prove a certain generalization of (1.4) without the a priory condition $u(0) = 0$ (Theorem 2.1). For Schwarz lemma for the derivatives of harmonic mappings on the plane and space we refer to the papers [7, 6]. It is worth to mention here a certain generalization of (1.2) for the mappings which are solution of certain elliptic partial differential equations in the plane [2]. For certain boundary Schwarz lemma on the unit ball for holomorphic mappings in $\mathbf{C}^n$ we refer to the paper [8].

By using Hopf theorem it can be proved ([5]) that if u is a harmonic mapping of the unit ball onto itself such that $u(0) = 0$ and $\|u(\zeta)\| = 1$, then

$$\liminf_{r\to 1} \left\|\frac{\partial u}{\partial r}(r\zeta)\right\| \geqslant C_n,$$

where C_n is a certain positive constant. Our goal is to find the largest constant C_n. This is done in Theorem 2.3 and Theorem 2.4.

2. Preliminaries and main results

First we prove the following generalization of harmonic Schwarz lemma for B^n, $n \geqslant 3$. The case $n = 2$ has been treated and proved by Pavlovic [9, Theorem 3.6.1].

Theorem 2.1. *Let u be a harmonic mapping of the unit ball onto itself, then*

$$\left\|u(x) - \frac{1-\|x\|^2}{(1+\|x\|^2)^{n/2}} u(0)\right\| \leqslant U(\|x\|N). \tag{2.1}$$

Proof. Assume first that $x = rN$. We have that

$$u(rN) = \int_{S^{n-1}} \frac{1-r^2}{\|\zeta - rN\|^n} f(\zeta) d\sigma(\zeta),$$

and so

$$u(rN) - \frac{1-r^2}{(1+r^2)^{n/2}} u(0) = \int_{S^{n-1}} \left(\frac{1-r^2}{\|\zeta - rN\|^n} - \frac{1-r^2}{(1+r^2)^{n/2}} \right) f(\zeta) d\sigma(\zeta).$$

Further we have

$$\begin{aligned} \|u(rN) - \frac{1-r^2}{(1+r^2)^{n/2}} u(0)\| &\leqslant \int_{S^{n-1}} \left| \frac{1-r^2}{\|\zeta - rN\|^n} - \frac{1-r^2}{(1+r^2)^{n/2}} \right| d\sigma(\zeta) \\ &= \int_{S^+} \left(\frac{1-r^2}{\|\zeta - rN\|^n} - \frac{1-r^2}{(1+r^2)^{n/2}} \right) d\sigma(\zeta) \\ &+ \int_{S^-} \left(\frac{1-r^2}{(1+r^2)^{n/2}} - \frac{1-r^2}{\|\zeta - rN\|^n} \right) d\sigma(\zeta). \end{aligned}$$

Thus

$$\left\| u(rN) - \frac{1-r^2}{(1+r^2)^{n/2}} u(0) \right\| \leqslant U(rN).$$

Now if x is not on the ray $[0, N]$, we choose a unitary transformation O such that $O(N) = x/|x|$. Then we make use of harmonic mapping $v(y) = u(O(y))$ for which we have $v(rN) = u(O(rN)) = u(x)$. By making use of the previous proof we obtain (2.1). □

2.1. **Hypergeometric functions.** In order to formulate and to prove our next results recall the basic definition of hypergeometric functions. For two positive integers p and q and vectors $a = (a_1, \dots, a_p)$ and $b = (b_1, \dots, b_q)$ we set

$${}_pF_q[a; b, x] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k \cdot k!} x^k,$$

where $(y)_k := \frac{\Gamma(y+k)}{\Gamma(y)} = y(y+1)\dots(y+k-1)$ is the Pochhammer symbol. The hypergeometric series converges at least for $|x| < 1$. For basic properties and formulas concerning trigonometric series we refer to the book [3]. The most important step in the proof of our main results i.e. of Theorem 2.3 and Theorem 2.4 below, is the following lemma

Lemma 2.2. *The function $V(r) = \frac{\partial U(rN)}{\partial r}$, $0 \leqslant r \leqslant 1$ is decreasing on the interval $[0, 1]$ and we have*

$$V(r) \geqslant V(1) = C_n := \frac{n! \left(1 + n - (n-2)\, {}_2\mathrm{F}_1 \left[\frac{1}{2}, 1, \frac{3+n}{2}, -1\right]\right)}{2^{3n/2} \Gamma\left[\frac{1+n}{2}\right] \Gamma\left[\frac{3+n}{2}\right]}.$$

Proof. By using spherical coordinates $\eta=(\eta_1,\dots,\eta_n)$ such that $\eta_n=\cos\theta$, where θ is the angle between the vector x and x_n axis, we obtain from (1.5) that

$$U(rN)=\frac{\Gamma\left[\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi}\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}(\chi_{S^+}(x)-\chi_{S^-}(x))d\theta$$

and so

$$U(rN)=\frac{\Gamma\left[\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi/2}\left(\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\cos^{n-2}\theta}{(1+r^2+2r\sin\theta)^{n/2}}\right)d\theta$$

or what can be written as

$$U(rN)=\frac{\Gamma\left[\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\int_0^{\pi/2}\left(\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2+2r\cos\theta)^{n/2}}\right)d\theta.$$

Let $P=2r/(1+r^2)$. Then

$$\begin{aligned}&\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2-2r\cos\theta)^{n/2}}-\frac{(1-r^2)\sin^{n-2}\theta}{(1+r^2+2r\cos\theta)^{n/2}}\\&=\frac{(1-r^2)}{(1+r^2)^{n/2}}\sum_{k=0}^{\infty}\left(\binom{-n/2}{k}((-1)^k-1)\cos^k\theta\sin^{n-2}\theta\right)P^k.\end{aligned}$$

Since

$$\int_0^{\pi/2}\cos^k\theta\sin^{n-2}\theta d\theta=\frac{\Gamma\left[\frac{1+k}{2}\right]\Gamma\left[\frac{1}{2}(-1+n)\right]}{2\Gamma\left[\frac{k+n}{2}\right]},$$

we obtain

$$U(rN)=\frac{\Gamma\left[\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n-1}{2}\right]}\frac{(1-r^2)}{(1+r^2)^{n/2}}\sum_{k=0}^{\infty}\frac{\Gamma\left[\frac{1+k}{2}\right]\Gamma\left[\frac{n-1}{2}\right]}{2\Gamma\left[\frac{k+n}{2}\right]}\binom{-n/2}{k}((-1)^k-1)P^k.$$

Hence

$$U(rN)=r\left(1-r^2\right)\left(1+r^2\right)^{-1-\frac{n}{2}}\frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}G(r),$$

where

$$G(r)={}_3\mathrm{F}_2\left[1,\frac{2+n}{4},\frac{4+n}{4};\frac{3}{2},\frac{1+n}{2};\frac{4r^2}{\left(1+r^2\right)^2}\right].$$

By [3, Eq. 3.1.8] for $a=\frac{n}{2},\ b=\frac{1}{2}(-1+n),\ c=\frac{1}{2}$, we have that

$$G(r)=\frac{\left(1+r^2\right)^{1+\frac{n}{2}}{}_4F_3\left[\left\{\frac{n}{2},\frac{1}{2}(-1+n),\frac{1}{2},1+\frac{n}{4}\right\},\left\{\frac{n}{4},\frac{3}{2},\frac{1}{2}+\frac{n}{2}\right\},-r^2\right]}{1-r^2}.$$

So

$$U(rN)=r\frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}{}_4F_3\left[\left\{\frac{n}{2},\frac{1}{2}(-1+n),\frac{1}{2},1+\frac{n}{4}\right\},\left\{\frac{n}{4},\frac{3}{2},\frac{1}{2}+\frac{n}{2}\right\},-r^2\right],$$

which can be written as

$$U(rN)=\frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}r+\sum_{k=1}^{\infty}\frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(1+2k)(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]}r^{2k+1}.$$

Thus

$$\frac{\partial U(rN)}{\partial r}=\frac{2\Gamma\left[1+\frac{n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{1+n}{2}\right]}+\sum_{k=1}^{\infty}\frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]}r^{2k}.$$

Since

$$\begin{aligned}&\frac{2(-1)^k(4k+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[1+k]\Gamma\left[\frac{1}{2}(n-1)\right]}\\&=\frac{(-1)^k2^n\Gamma\left[1+\frac{n}{2}\right]\Gamma\left[k+\frac{n}{2}\right]}{\pi k!\Gamma[n]}+\frac{2(-1)^k(-2+n)\Gamma\left[k+\frac{n}{2}\right]}{(-1+2k+n)\sqrt{\pi}\Gamma[k]\Gamma\left[\frac{1+n}{2}\right]}\end{aligned}$$

we obtain that

$$\frac{\partial U(rN)}{\partial r}=\frac{\Gamma\left[1+\frac{n}{2}\right]\left((1+r^2)^{-n/2}(1+n)-(n-2)r^2\,{}_2F_1\left[\frac{1+n}{2},\frac{2+n}{2},\frac{3+n}{2},-r^2\right]\right)}{\sqrt{\pi}\Gamma\left[\frac{3+n}{2}\right]},$$

which in view of the Kummer quadratic transformation, can be written in the form

$$\frac{\partial U(rN)}{\partial r}=\frac{\Gamma\left[1+\frac{n}{2}\right](1+r^2)^{-n/2}\left(1+n-(n-2)r^2\,{}_2F_1\left[\frac{1}{2},1,\frac{3+n}{2},-r^2\right]\right)}{\sqrt{\pi}\Gamma\left[\frac{3+n}{2}\right]}.$$

The function

$$y{}_2F_1[1/2,1,(3+n)/2,-y]$$

increases in y. Namely its derivative is

$$\begin{aligned}{}_2F_1[1/2,2,(3+n)/2,-y]&=\sum_{m=0}^{\infty}(-1)^m a(m)y^m\\&=\sum_{m=0}^{\infty}\frac{(-1)^m(1+m)\Gamma\left[\frac{1}{2}+m\right]\Gamma\left[\frac{3+n}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{3}{2}+m+\frac{n}{2}\right]}y^m.\end{aligned}$$

Then $a(m)>0$ and

$$\frac{a(m)}{a(m+1)}=\frac{(1+m)(3+2m+n)}{(2+m)(1+2m)}>1$$

because $1+n+mn>0$, and so

$${}_2F_1[1/2,2,(3+n)/2,-y]\geqslant\sum_{m=0}^{\infty}(a(2m)-a(2m+1))y^{2m}>0.$$

The conclusion is that $\frac{\partial U(rN)}{\partial r}$ is decreasing. In particular

$$\frac{\partial U(rN)}{\partial r}\geqslant\frac{\partial U(rN)}{\partial r}|_{r=1}.$$

For $r=1$ we have
$$\frac{\partial U(rN)}{\partial r}=C_n=\frac{n!\left(1+n-(n-2)\,{}_2\mathrm{F}_1\left[\frac{1}{2},1,\frac{3+n}{2},-1\right]\right)}{2^{3n/2}\Gamma\left[\frac{1+n}{2}\right]\Gamma\left[\frac{3+n}{2}\right]}.$$
□

Theorem 2.3. *If u is a harmonic mapping of the unit ball into itself such that $u(0)=0$, then for $x\in B$ the following sharp inequality*
$$\frac{1-\|u(x)\|}{1-\|x\|}\geqslant C_n$$
holds.

Proof. From Theorem 2.1 we have that $\|u(x)\|\leqslant U(rN)$ and so
$$\frac{1-\|u(x)\|}{1-\|x\|}\geqslant\frac{1-|U(rN)|}{1-\|x\|}.$$

Further there is $\rho\in(r,1)$ such that
$$\frac{1-U(rN)}{1-\|x\|}=\frac{\partial U(\rho N)}{\partial r},$$
which in view of Lemma 2.2 is bigger that C_n. The proof is completed. □

Theorem 2.4. *a) If u is a harmonic mapping of the unit ball* **into** *itself such that $u(0)=0$, and for some $\|\zeta\|=1$ we have $\lim_{r\to 1}\|u(r\zeta)\|=1$ then*
$$\liminf_{r\to 1^-}\left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\|\geqslant C_n. \tag{2.2}$$

b) If u is a proper harmonic mapping of the unit ball **onto** *itself such that $u(0)=0$, then the following sharp inequality*
$$\liminf_{r\to 1^-}\left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\|\geqslant C_n,\quad \|\zeta\|=1 \tag{2.3}$$
holds. Here and in the sequel $\mathbf{n}$ *is outward-pointing unit normal.*

Proof. Prove a). Then b) follows from a). Let $0<r<1$ and $x\in(r\zeta,\zeta)$. There is a $\rho\in(\|x\|,1)$ such that
$$\frac{1-\|u(x)\|}{1-r}=\left.\frac{\partial\|u(r\zeta)\|}{\partial r}\right|_{r=\rho}. \tag{2.4}$$
On the other hand
$$\left\|\frac{\partial u(r\zeta)}{\partial r}\right\|\geqslant\frac{\partial\|u(r\zeta)\|}{\partial r}.$$
Letting $\|x\|=r\to 1$, in view of Thereom 2.3 and (2.4), we obtain that
$$\liminf_{r\to 1}\left\|\frac{\partial u}{\partial \mathbf{n}}(r\zeta)\right\|\geqslant C_n.$$

To show that the inequality (2.3) is sharp, let

$$h_m(x) = \begin{cases} 1 - x/m, & \text{if } x \in (1/m, 1]; \\ (m-1)x, & \text{if } -1/m \leqslant x \leqslant 1/m; \\ -1 - x/m, & \text{if } x \in [-1, -1/m), \end{cases}$$

and define

$$f_m(x_1, \dots, x_{n-1}, x_n) = \frac{\sqrt{1 - h_m(x_n)^2}}{\sqrt{1 - x_n^2}}(x_1, \dots, x_{n-1}, 0) + (0, \dots, 0, h_m(x_n)).$$

Then f_m is a homeomorphism of the unit sphere onto itself, such that

$$\lim_{m\to\infty} f_m(x) = (0, \dots, 0, \chi_{S^+}(x) - \chi_{S^-}(x)).$$

Further $u_m(x) = \mathcal{P}[f_m](x)$ is a harmonic mapping of the unit ball onto itself such that $\lim_{\|x\|\to 1} \|u_m(x)\| = 1$. Thus u_m is proper. Moreover $u_m(0) = 0$ and $\lim_{m\to\infty} u_m(x) = (0, \dots, 0, U(x))$. This implies the fact that the constant C_n is sharp. □

Remark 2.5. The following table shows first few constants C_n and related functions

n	$u(rN)$	$\partial_r u(rN)$	C_n
2	$\frac{4\arctan(r)}{\pi}$	$\frac{4}{\pi(1+r^2)}$,	$\frac{2}{\pi}$
3	$\frac{-1+r^2+\sqrt{1+r^2}}{r\sqrt{1+r^2}}$	$\frac{1-\sqrt{1+r^2}-r^2\left(-3+\sqrt{1+r^2}\right)}{r^2(1+r^2)^{3/2}}$	$\sqrt{2}-1$
4	$\frac{2r\left(-1+r^2\right)+2\left(1+r^2\right)^2\arctan r}{\pi r^2(1+r^2)}$	$\frac{4\left(r+3r^3-\left(1+r^2\right)^2\arctan r\right)}{\pi r^3(1+r^2)^2}$	$\frac{4-\pi}{\pi}$

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